

# A mystery of black-hole gravitational resonances

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## Abstract

More than three decades ago, Detweiler provided an analytical formula for the gravitational resonant frequencies of rapidly-rotating Kerr black holes. In the present work we shall discuss an important discrepancy between the famous *analytical* prediction of Detweiler and the recent *numerical* results of Zimmerman et. al. In addition, we shall refute the claim that recently appeared in the physics literature that the Detweiler-Teukolsky-Press resonance equation for the characteristic gravitational eigenfrequencies of rapidly-rotating Kerr black holes is not valid in the regime of damped quasinormal resonances with  $\Im\omega/T_{\text{BH}} \gg 1$  (here  $\omega$  and  $T_{\text{BH}}$  are respectively the characteristic quasinormal resonant frequency of the Kerr black hole and its Bekenstein-Hawking temperature). The main goal of the present paper is to highlight and expose this important *black-hole quasinormal mystery* (that is, the intriguing discrepancy between the analytical and numerical results regarding the gravitational quasinormal resonance spectra of rapidly-rotating Kerr black holes).

## I. THE BLACK-HOLE QUASINORMAL MYSTERY

One of the earliest and most influential works on the quasinormal resonance spectra of black holes is Detweiler's "Black holes and gravitational waves. III. The resonant frequencies of rotating holes" [1]. In this famous paper [2], he derived the characteristic resonance equation

$$-\frac{\Gamma(2i\delta)\Gamma(1+2i\delta)\Gamma(1/2+s-i\hat{\omega}-i\delta)\Gamma(1/2-s-i\hat{\omega}-i\delta)}{\Gamma(-2i\delta)\Gamma(1-2i\delta)\Gamma(1/2+s-i\hat{\omega}+i\delta)\Gamma(1/2-s-i\hat{\omega}+i\delta)} = (-i\hat{\omega}\tau)^{2i\delta} \frac{\Gamma(1/2+i\hat{\omega}+i\delta-4i\varpi/\tau)}{\Gamma(1/2+i\hat{\omega}-i\delta-4i\varpi/\tau)} \quad (1)$$

for the quasinormal frequencies of near-extremal (rapidly-rotating) Kerr black holes. Here [3, 4]

$$\tau \equiv 8\pi MT_{\text{BH}} \quad ; \quad \varpi \equiv M(\omega - m\Omega_{\text{H}}) \quad ; \quad \hat{\omega} \equiv 2\omega r_+ , \quad (2)$$

where

$$T_{\text{BH}} = \frac{r_+ - r_-}{8\pi M r_+} \quad ; \quad \Omega_{\text{H}} = \frac{a}{2Mr_+} \quad (3)$$

are the Bekenstein-Hawking temperature and the angular velocity of the rotating Kerr black hole, respectively. The parameters  $\{s, m\}$  are the spin-weight and azimuthal harmonic index of the field mode [5], and  $\delta$  is closely related to the angular-eigenvalue of the angular Teukolsky equation [5, 6].

Detweiler's resonance equation (1) is based on the earlier analyzes of Teukolsky and Press [5] and Starobinsky and Churilov [7] who studied the scattering of massless spin- $s$  perturbation fields in the rotating Kerr black-hole spacetime in the double limit  $a/M \rightarrow 1$  ( $T_{\text{BH}} \rightarrow 0$ ) and  $\omega \rightarrow m\Omega_{\text{H}}$ . These limits correspond to [see Eq. (2)]

$$\tau \rightarrow 0 \quad \text{and} \quad \varpi \rightarrow 0 . \quad (4)$$

The rather complicated resonance equation (1) can be solved *analytically* in two distinct regimes:

(1) In his original analysis [1], Detweiler studied the regime

$$\frac{\varpi}{\tau} \gg 1 \quad \text{with} \quad \delta^2 > 0 , \quad (5)$$

and obtained the black-hole eigenfrequencies

$$\varpi_n = -\frac{e^{\theta/2\delta}}{4m} (\cos \phi + i \sin \phi) \times e^{-\pi n/\delta} , \quad (6)$$

where the integer  $n$  is the resonance parameter of the mode, and

$$re^{i\theta} \equiv \left[ \frac{\Gamma(2i\delta)}{\Gamma(-2i\delta)} \right]^2 \frac{\Gamma(1/2 + s - im - i\delta)\Gamma(1/2 - s - im - i\delta)}{\Gamma(1/2 + s - im + i\delta)\Gamma(1/2 - s - im + i\delta)} \quad ; \quad \phi \equiv -\frac{1}{2\delta} \ln r . \quad (7)$$

(2) On the other hand, in [8] (see also [9]) we have analyzed the regime

$$\frac{\varpi}{\tau} = O(1) , \quad (8)$$

and obtained the characteristic relation [8]

$$\Im \varpi_n = -2\pi T_{\text{BH}} \left( n + \frac{1}{2} + \Im \delta \right) \quad (9)$$

for the *slowly* damped quasinormal resonances of the rapidly-rotating (near-extremal,  $T_{\text{BH}} \rightarrow 0$ ) Kerr black holes.

Recently, Yang. et. al. [10] have used numerical techniques to compute the quasinormal resonances of rapidly-rotating Kerr black holes. They have found numerically [10] that the slowly damped quasinormal resonances of these near-extremal black holes are described extremely well by the analytical formula (9) of [8, 9]. On the other hand, in their numerical study, Yang. et. al. [10] have found *no* trace of the black-hole quasinormal resonances (6) predicted by Detweiler. This discrepancy between the *analytical* prediction (6) of [1] and the *numerical* results of [10] is the essence of the black-hole quasinormal mystery.

Most recently, Zimmerman et. al. [11] have claimed that the discrepancy between Detweiler's *analytical* prediction (6) and their *numerical* results [10, 11] stems from the fact that his resonance equation (1) is not valid in the regime (5). In particular, they have claimed that the standard matching procedure used in [5, 7] to match the near-horizon

$$x \equiv \frac{r - r_+}{r_+} \ll 1 \quad (10)$$

solution [see equation (A9) of [5]]

$$R = {}_2F_1(-i\hat{\omega} + s + 1/2 + i\delta, -i\hat{\omega} + s + 1/2 - i\delta; 1 + s - 4i\varpi/\tau; -x/\tau) \quad (11)$$

of the Teukolsky radial equation with the far-region  $x \gg \max(\varpi, \tau)$  solution [see equation (A5) of [5]] of the Teukolsky radial equation is invalid in the regime (5) studied by Detweiler. Since Detweiler's analysis is based on the matching procedure of [5, 7], Zimmerman et. al. have claimed that Detweiler's analysis is also invalid in the regime (5).

## II. THE MYSTERY IS STILL UNSOLVED

In this paper we would like to point out that the assertion made in Ref. [11] is actually erroneous. In particular, we shall show below that the matching procedure of [5, 7, 12] is valid in the overlap region

$$\max(\varpi, \tau) \ll x \ll 1. \quad (12)$$

In their matching procedure, Teukolsky and Press [5] (see also [7]) use the identity [see Eq. 15.3.7 of [13]]

$$R = \frac{\Gamma(1+s-4i\varpi/\tau)\Gamma(2i\delta)}{\Gamma(-i\hat{\omega}+s+1/2+i\delta)\Gamma(1/2+i\hat{\omega}+i\delta-4i\varpi/\tau)} \left(\frac{x}{\tau}\right)^{i\hat{\omega}-s-1/2+i\delta} \quad (13)$$

$$\times {}_2F_1(-i\hat{\omega}+s+1/2-i\delta, -i\hat{\omega}+1/2-i\delta+4i\varpi/\tau; 1-2i\delta; -\tau/x) + (\delta \rightarrow -\delta)$$

for the near-horizon hypergeometric function (11) [The notation  $(\delta \rightarrow -\delta)$  in (13) means “replace  $\delta$  by  $-\delta$  in the preceding term.”]. In order to perform the matching procedure, Teukolsky and Press [5] (see also [7]) take the limit

$${}_2F_1(-i\hat{\omega}+s+1/2\mp i\delta, -i\hat{\omega}+1/2\mp i\delta+4i\varpi/\tau; 1\mp 2i\delta; -\tau/x) \rightarrow 1 \quad (14)$$

for the hypergeometric functions that appear in the expression (13) of the radial eigenfunction  $R$ .

In Ref. [11] Zimmerman et. al. have recently claimed that the limit (14) used in [5, 7, 12] is invalid in the regime (5) studied by Detweiler. To support their claim, they plot (see Fig. 3 of [11]) the hypergeometric functions of (13) in the regime

$$\frac{\varpi}{x} = 250. \quad (15)$$

Not surprisingly, Zimmerman et. al. found that, in the regime (15), the asymptotic behavior (14) used in the matching procedure of [5, 7, 12] is not valid. They then concluded that the matching procedure of [5] is not valid in the regime (5) studied by Detweiler.

However, here we would like to stress the fact that the analytical arguments of Zimmerman et. al. (and, in particular, the data presented in Fig. 3 of [11]) are actually *irrelevant* for the discussion about the validity of Detweiler’s resonance condition (1). In particular, we would like to emphasize the fact that the characteristic limiting behavior

$${}_2F_1(a, b; c; z) \rightarrow 1 \quad (16)$$

of the hypergeometric function is valid in the regime [14]

$$\frac{a \cdot b \cdot z}{c} \ll 1 . \quad (17)$$

Taking cognizance of the arguments  $(a, b, c, z)$  of the hypergeometric functions in (13), one realizes that, for moderate values of the field azimuthal harmonic index  $m$ , the asymptotic behavior (14) assumed in the matching procedure of [5, 7, 12] is valid in the regime [see Eqs. (13) and (17)]

$$\max\left(1, \frac{\varpi}{\tau}\right) \times \frac{\tau}{x} \ll 1 . \quad (18)$$

In particular, one finds from (18) that, in the regime  $\varpi/\tau \gg 1$  [see (5)] studied by Detweiler, the asymptotic behavior (14) used in [5, 7, 12] is valid for

$$\frac{\varpi}{x} \ll 1 . \quad (19)$$

This is certainly *not* the regime plotted in Fig. 3 of [11] [see Eq. (15)]. One therefore concludes that the analytical arguments raised by Zimmerman et. al. (and, in particular, the data presented in Fig. 3 of [11]) are simply irrelevant for the discussion about the validity of the matching procedure used in [5, 7]!

Furthermore, taking cognizance of (10) and (18), one concludes that the matching procedure of [5, 7] is *valid* in the overlap region

$$\max\left(1, \frac{\varpi}{\tau}\right) \times \tau \ll x \ll 1 . \quad (20)$$

It is worth emphasizing again that Detweiler's analysis is based on the matching procedure of [5, 7]. As such, his resonance equation (1) for the characteristic eigen-frequencies of rapidly-rotating Kerr black holes is expected to be valid in the regime (20) [and *not* in the regime (15) considered in Fig. 3 of [11]].

In Table I we present the hypergeometric functions (13) used in the matching procedure of [5, 7]. We display the values of these functions for the field mode  $l = m = s = 2$  with  $n = 2, 3, 4, 5$  and  $x = 0.1$  [see (10)]. This mode is characterized by the angular eigenvalue  $\delta = 2.051$  [15], which yields the Detweiler resonance spectrum [see Eqs. (6) and (7)]

$$\varpi_n \simeq (0.162 - 0.035i) \times e^{-1.532n} . \quad (21)$$

Inspection of the data presented in Table I reveals that, for resonant modes with  $n \geq 3$ , the hypergeometric functions  ${}_2F_1$  that appear in the expression (13) for the radial eigenfunction  $R$  are described extremely well by the asymptotic behavior (14) as originally assumed

in [5, 7, 12]. Our results therefore support the validity of the matching procedure used by Teukolsky and Press [5] and by Starobinsky and Churilov [7].

$\tau$	$10^{-4}$	$10^{-5}$	$10^{-6}$
$n = 2$	$0.972 - 0.344i$	$0.972 - 0.348i$	$0.972 - 0.349i$
	$0.836 - 0.003i$	$0.836 - 0.003i$	$0.836 - 0.003i$
$n = 3$	$1.003 - 0.071i$	$1.005 - 0.075i$	$1.005 - 0.075i$
	$0.961 - 0.001i$	$0.961 - 0.002i$	$0.961 - 0.002i$
$n = 4$	$0.999 - 0.012i$	$1.001 - 0.016i$	$1.001 - 0.016i$
	$0.991 - 0.000i$	$0.991 - 0.000i$	$0.991 - 0.000i$
$n = 5$	$0.998 + 0.001i$	$1.000 - 0.003i$	$1.000 - 0.003i$
	$0.998 + 0.000i$	$0.998 - 0.000i$	$0.998 - 0.000i$

TABLE I: The values of the hypergeometric functions (13) used in the matching procedure of [5, 7] for the field mode  $l = m = s = 2$  with  $x = 0.1$ . The first row corresponds to the hypergeometric function  ${}_2F_1(-i\hat{\omega} + s + 1/2 - i\delta, -i\hat{\omega} + 1/2 - i\delta + 4i\varpi/\tau; 1 - 2i\delta; -\tau/x)$  in (13), whereas the second row corresponds to the hypergeometric function  ${}_2F_1(-i\hat{\omega} + s + 1/2 + i\delta, -i\hat{\omega} + 1/2 + i\delta + 4i\varpi/\tau; 1 + 2i\delta; -\tau/x)$  in (13). One finds that, for resonant modes with  $n \geq 3$ , these functions are described extremely well by the asymptotic behavior (14).

### III. SUMMARY

Currently, we face a mystery which can be summarized as follows:

- (1) Contrary to the claim made in [11], Detweiler's resonance equation (1) is valid in the regime (5).
- (2) In his original *analytical* study, Detweiler has shown that the resonance equation (1) predicts the existence of the near-extremal Kerr black-hole quasinormal resonances (6) in the regime (5).
- (3) In their recent *numerical* study, Zimmerman et. al. have found *no* trace of the black-hole quasinormal resonances (6) predicted by Detweiler.

The main goal of the present work was to highlight the discrepancy between the *analytical* prediction (6) of Detweiler and the *numerical* results of Zimmerman et. al. We would like

to emphasize that currently we have no solution to this black-hole quasinormal mystery. We believe, however, that it is important to highlight this open problem. We hope, in particular, that the present work would encourage researchers in the fields of black-hole physics and general relativity to further explore this interesting physical problem. Hopefully, future studies of the black-hole quasinormal spectrum will shed light on this intriguing mystery.

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  - [4] Here  $M, a \equiv J/M$ , and  $r_{\pm} = M + (M^2 - a^2)^{1/2}$  are the mass, angular momentum per unit mass, and horizon radii of the rotating Kerr black hole, respectively.
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